

Dichotomy results for norm estimates in operator semigroups

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Dedicated to Charles Batty on the occasion of his sixtieth birthday.

*In line with an Oxford tradition,
Charles Batty was sent on a mission:
to instruct all the troops
in C_0 semigroups -
this indeed was a brilliant decision!*

1 Introduction

We recall that a one-parameter family $(T(t))_{0 \leq t < \infty}$ in a Banach algebra (often itself simply the algebra of bounded linear operators on a Banach space \mathcal{X}) is a semigroup if

$$T(s+t) = T(s)T(t) \quad \text{for all } t, s > 0.$$

We shall be concerned here with semigroups that are strongly continuous on $\mathbb{R}_+ := (0, \infty)$, but not necessarily norm-continuous at the origin. As an

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example to bear in mind we mention the semigroup $T(t) : x \mapsto x^t$ in the algebra $C_0([0, 1])$ of continuous functions on $[0, 1]$ vanishing at the origin, which will be discussed later.

Later, we consider semigroups defined on a sector in the complex plane, in which case they will be assumed to be *analytic*: that is, complex-differentiable in the norm topology.

The results in this survey indicate that the quantitative behaviour of the semigroup at the origin provides additional qualitative information, such as uniform continuity or analyticity. Here are a few examples.

We recall the classical *zero-one law*, asserting that if $\limsup_{t \rightarrow 0^+} \|T(t) - I\| < 1$, then in fact $\|T(t) - I\| \rightarrow 0$ and hence the semigroup is uniformly continuous, and of the form e^{tA} for some bounded operator A . To see this, set $L = \limsup_{t \rightarrow 0^+} \|T(t) - I\|$. Since

$$2(T(t) - I) = T(2t) - I - (T(t) - I)^2,$$

we have $2L \leq L + L^2$, and thus $L = 0$ or $L \geq 1$.

Another result involving the asymptotic behaviour at 0 and providing a uniformly continuous semigroup is the following, proved in 1950 by Hille [12] (see also [13, Thm. 10.3.6]). This result is usually stated for $n = 1$, but Hille's argument works for any positive integer.

Theorem 1.1 *Let $(T(t))_{t \geq 0}$ be a n -times continuously differentiable semigroup over the positive reals. If $\limsup_{t \rightarrow 0^+} \|t^n T^{(n)}(t)\| < (\frac{n}{e})^n$, then the generator of the semigroup is bounded.*

In the direction of analyticity, a classical result of Beurling [3] is the following:

Theorem 1.2 *A C_0 -semigroup $(T(t))_{0 \leq t < \infty}$ on a complex Banach space \mathcal{X} is holomorphic if and only if there exists a polynomial p such that*

$$\limsup_{t \rightarrow 0^+} \|p(T(t))\| < \sup\{|p(z)| : |z| \leq 1\}. \quad (1)$$

Kato [15] and Neuberger [18] proved the sufficiency of (1) with $p(z) = z - 1$, and $\sup |p(z)| = 2$, providing a zero–two law for analyticity. In general the converse is not true with $p(z) = z - 1$, although it holds if \mathcal{X} is uniformly convex and the semigroup is contractive [19]. Some extensions of this result to arbitrary Banach spaces and for semigroups which are not necessarily contractive may be found in the very recent paper [11].

The more recent results considered in this survey concern estimates of the norm or spectral radius of quantities such as $T(t) - T((n+1)t)$ as t tends to 0. These are often formulated as dichotomy results, such as the zero–quarter law (the case $n = 1$ in the following theorem).

Theorem 1.3 ([10, 17]) *Let $n \geq 1$ be an integer, and let $(T(t))_{t \geq 0}$ be a semigroup in a Banach algebra. If*

$$\limsup_{t \rightarrow 0^+} \|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+1/n}},$$

then either $T(t) = 0$ for $t > 0$ or else the closed subalgebra generated by $(T(t))_{t \geq 0}$ is unital, and the semigroup has a bounded generator A : that is, $T(t) = \exp(tA)$ for $t > 0$.

Another result that we mention here concerns the link between the norm and spectral radius ρ , and was motivated also by the Esterle–Katznelson–Tzafriri results on estimates for $\|T^n - T^{n+1}\|$, where T is a power-bounded operator (see [8, 16]). We have rewritten it in the notation of differentiable groups $(T(t)) = (\exp(tA))$, which may even be defined for $t \in \mathbb{C}$ if A is bounded; note that $T'(t) = AT(t)$.

Theorem 1.4 ([14]) *Let A be a bounded operator on a Banach space, and let $(T(t))$ be the group given by $T(t) = \exp(tA)$. Then each of the following conditions implies that $\rho(A) = \|A\|$.*

- (i) $\sup_{t>0} t\|T'(t)\| \leq 1/e$;
- (ii) $\sup_{t>0} \|T(t) - T((s+1)t)\| \leq s(s+1)^{-(1+1/s)}$ for some $s > 0$;
- (iii) $\sup_{t>0} \|T((s+i)t) - T((s-i)t)\| \leq 2e^{-s \arctan(1/s)}/\sqrt{1+s^2}$ for some $s \geq 0$.

The third condition of Theorem 1.4 is linked to the Bonsall–Crabb proof of Sinclair’s spectral radius formula for Hermitian elements of a Banach algebra, given in [4].

In Section 2 we review the existing literature on dichotomy laws for semigroups, first for semigroups on \mathbb{R}_+ and then for analytic semigroups defined on a sector in the complex plane; our main sources here are [1] and [5]. Then in Section 3 we present some very recent generalizations of these results, formulated in the language of functional calculus: this discussion is based on [6] and [7].

2 Dichotomy laws

2.1 Semigroups on \mathbb{R}_+

We begin with a result on quasinilpotent semigroups, that is, semigroups whose elements all have spectral radius 0. This is a commonly-occurring case, examples being found in the convolution algebra $L^1(0, 1)$.

Theorem 2.1 ([9]) *Let $(T(t))_{t>0}$ be a C_0 -semigroup of bounded quasinilpotent linear operators on a Banach space \mathcal{X} . Then there exists $\delta > 0$ such that*

$$\|T(t) - T(s)\| > \theta(s, t) \quad \text{for} \quad 0 < t < s < \delta,$$

where

$$\theta(s, t) = (s - t)t^{t/(s-t)}s^{s/(t-s)}.$$

In particular, for all $\gamma > 0$, there exists $\delta > 0$ such that

$$\|T(t) - T((\gamma + 1)t)\| > \frac{\gamma}{(\gamma + 1)^{1+1/\gamma}}$$

for all $0 < t < \delta$.

This is a sharp result, in the sense that given a non-decreasing function $\epsilon : (0, 1) \rightarrow (0, \infty)$ there exists a nontrivial quasinilpotent semigroup $(T_\epsilon(t))_{t>0}$ on a Hilbert space such that:

$$\|T_\epsilon(t) - T_\epsilon(s)\| \leq \theta(s, t) + (s - t)\epsilon(s)$$

(see [9]). Note that $\theta(s, t) = \max_{0 \leq x \leq 1} (x^t - x^s)$.

It is a quantitative formulation of an intuitive fact: $T(t)$ cannot be uniformly too close to $T(s)$ for $s \neq t$, with s, t small when the generator is unbounded.

In the non-quasinilpotent case, it is possible to formulate similar results using the spectral radius. The following theorem is a strengthening of [10, Thm 2.3], which is expressed in terms of $\limsup_{t \rightarrow 0} \rho(T(t) - T(t(\gamma + 1)))$. Note that $\text{Rad } \mathcal{A}$ denotes the radical of the algebra \mathcal{A} , i.e., the set of elements with spectral radius zero.

Theorem 2.2 ([1]) *Let $(T(t))_{t>0}$ be a non-quasinilpotent semigroup in a Banach algebra, let \mathcal{A} be the closed subalgebra generated by $(T(t))_{t>0}$, and let $\gamma > 0$ be a real number. If there exists $t_0 > 0$ such that*

$$\rho(T(t) - T(t(\gamma + 1))) < \frac{\gamma}{(\gamma + 1)^{1+\frac{1}{\gamma}}}$$

for $0 < t \leq t_0$, then $\mathcal{A}/\text{Rad}(\mathcal{A})$ is unital, and there exist an idempotent J in \mathcal{A} , an element u of $J\mathcal{A}$ and a mapping $r : \mathbb{R}_+ \rightarrow \text{Rad}(J\mathcal{A})$, with the following

properties:

- (i) $\varphi(J) = 1$ for all $\varphi \in \widehat{\mathcal{A}}$;
- (ii) $r(s+t) = r(s) + r(t)$ for all $s, t \in \mathbb{R}_+$;
- (iii) $JT(t) = e^{tu+r(t)}$ for $t \in \mathbb{R}_+$, where $e^v = J + \sum_{k \geq 1} \frac{v^k}{k!}$ for $v \in J\mathcal{A}$;
- (iv) $(T(t) - JT(t))_{t \in \mathbb{R}_+}$ is a quasinilpotent semigroup.

If \mathcal{A} is semi-simple (that is, $\text{Rad}(\mathcal{A}) = \{0\}$), then the conclusion is much more straightforward.

Corollary 2.3 ([1]) *Let $(T(t))_{t>0}$ be a non-trivial semigroup in a commutative semi-simple Banach algebra, let \mathcal{A} be the closed subalgebra generated by $(T(t))_{t \in \mathbb{R}_+}$ and let $\gamma > 0$. If there exists $t_0 > 0$ such that*

$$\rho(T(t) - T((\gamma + 1)t)) < \frac{\gamma}{(\gamma + 1)^{1+\frac{1}{\gamma}}}$$

for $0 < t \leq t_0$, then \mathcal{A} is unital and there exists an element $u \in \mathcal{A}$ such that $T(t) = e^{tu}$ for $t \in \mathbb{R}_+$.

The following theorem needs no hypothesis on \mathcal{A} , but requires stronger estimates, based on the norm rather than the spectral radius.

Theorem 2.4 ([1]) *Let $(T(t))_{t>0}$ be a non-trivial semigroup in a Banach algebra, let \mathcal{A} be the closed subalgebra generated by $(T(t))_{t>0}$ and let $n \geq 1$ be an integer. If there exists $t_0 > 0$ such that*

$$\|(T(t) - T(t(n+1)))\| < \frac{n}{(n+1)^{1+\frac{1}{n}}}$$

for $0 < t \leq t_0$, then \mathcal{A} possesses a unit J , $\lim_{t \rightarrow 0^+} T(t) = J$ and there exists $u \in \mathcal{A}$ such that $T(t) = e^{tu}$ for all $t > 0$.

If $(T(t))_{t>0}$ is a quasinilpotent semigroup, then the condition

$$\|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+1/n}} \quad \text{for } 0 < t \leq t_0$$

implies that $T(t) = 0$ for all $t > 0$.

The sharpness of the above result is shown by the following example [1], which involves a construction of appropriate sequences in the non-unital Banach algebra c_0 .

Example 2.5 *Let G be an additive measurable subgroup of \mathbb{R} with $G \neq \mathbb{R}$. Then, given $(\gamma_n)_n$ in \mathbb{R}^+ such that $t\gamma_n \in G$ for all $t \in G$ with $t > 0$ and for all $n \in \mathbb{N}$, there exists a nontrivial semigroup $(S(t))_{t \in G, t > 0}$ in c_0 such that*

$$\|S(t) - S(t(\gamma_n + 1))\| < \frac{\gamma_n}{(\gamma_n + 1)^{1+1/\gamma_n}},$$

for all $t \in G$, $t > 0$.

2.2 Sectorial semigroups

In this subsection we discuss the behaviour of analytic semigroups defined on a sector

$$S_\alpha = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } |\arg(z)| < \alpha\}.$$

with $0 < \alpha \leq \pi/2$. We begin with the case $\alpha = \pi/2$ and $S_\alpha = \mathbb{C}_+$.

Theorem 2.6 ([1]) *Let $(T(t))_{t \in \mathbb{C}_+}$ be an analytic non-quasinilpotent semigroup in a Banach algebra. Let \mathcal{A} be the closed subalgebra generated by $(T(t))_{t \in \mathbb{C}_+}$ and let $\gamma > 0$. If there exists $t_0 > 0$ such that*

$$\sup_{t \in \mathbb{C}_+, |t| \leq t_0} \rho(T(t) - T(\gamma + 1)t) < 2$$

then $\mathcal{A}/\operatorname{Rad} \mathcal{A}$ is unital, and the generator of $(\pi(T(t)))_{t > 0}$ is bounded, where $\pi : \mathcal{A} \rightarrow \mathcal{A}/\operatorname{Rad} \mathcal{A}$ denotes the canonical surjection.

A semigroup $(T(t))$ defined on the positive reals or on a sector is said to be *exponentially bounded* if there exist $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that $\|T(t)\| \leq c_1 e^{c_2|t|}$ for every t . Beurling [3], in his work described in the introduction, showed that there exists a universal constant k such that every

exponentially bounded weakly measurable semigroup $(T(t))_{t>0}$ of bounded operators satisfying

$$\limsup_{t \rightarrow 0^+} \|I - T(t)\| = \rho < 2$$

admits an exponentially bounded analytic extension to a sector S_α with $\alpha \geq k(2 - \rho)^2$. From this one easily obtains the following result.

Theorem 2.7 ([1]) *Let $(T(t))_{t \in \mathbb{C}_+}$ be an analytic semigroup of bounded operators on a Banach space \mathcal{X} . If the generator of the semigroup is unbounded, then we have, for $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$,*

$$\limsup_{t \rightarrow 0^+} \|I - T(t)\| \geq 2 - \sqrt{\frac{\frac{\pi}{2} - |\alpha|}{k}},$$

where k is Beurling's universal constant.

We now consider similar results on smaller sectors than the half-plane, and in fact the result we prove will be stated in a far more general context.

Theorem 2.8 ([1]) *Let $0 < \alpha < \pi/2$ and let f be an entire function with $f(0) = 0$ and $f(\mathbb{R}) \subseteq \mathbb{R}$, such that*

$$\sup_{\operatorname{Re} z > r} |f(z)| \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad (2)$$

and f is a linear combination of functions of the form $z^m \exp(-zw)$ for $m = 0, 1, 2, \dots$ and $w > 0$. Let $(T(t))_{t \in S_\alpha} = (\exp(tA))_{t \in S_\alpha}$ be an analytic non-quasinilpotent semigroup in a Banach algebra and let \mathcal{A} be the subalgebra generated by $(T(t))_{t \in S_\alpha}$. If there exists $t_0 > 0$ such that

$$\sup_{t \in S_\alpha, |t| \leq t_0} \rho(f(-tA)) < k(S_\alpha),$$

with $k(S_\alpha) = \sup_{t \in S_\alpha} |f(z)|$, then $\mathcal{A}/\operatorname{Rad} \mathcal{A}$ is unital and the generator of $\pi(T(t))_{t \in S_\alpha}$ is bounded, where $\pi : \mathcal{A} \rightarrow \mathcal{A}/\operatorname{Rad}(\mathcal{A})$ denotes the canonical surjection.

Note that $f(-tA)$ is well-defined in terms of $T(t)$ and its derivatives.

Suitable examples of $f(z)$ are linear combinations of $z^m \exp(-z)$, $m = 1, 2, 3, \dots$, and $\exp(-z) - \exp(-(\gamma+1)z)$; also real linear combinations of the form $\sum_{k=1}^n a_k \exp(-b_k z)$ with $b_k > 0$ and $\sum_{k=1}^n a_k = 0$. This provides results analogous to those of [14, Thm. 4.12], where the behaviour of expressions such as $\|tA \exp(tA)\|$ and $\|\exp(tA) - \exp(stA)\|$ was considered for all $t > 0$.

Remark 2.9 *Another function considered in [14] is $f(z) = e^{-sz} \sin z$, where we now require $s > \tan \alpha$ for $f(-tA)$ to be well-defined for $t \in S_\alpha$. This does not satisfy the condition (2), but we note that it holds for $z \in S_\alpha$, while for $z \notin S_\alpha$ there exists a constant $C > 0$ with the following property: for each z with $\operatorname{Re} z > C$ there exists $\lambda \in (0, 1)$ such that $|f(\lambda z)| \geq \sup_{z \in S_\alpha} |f(z)|$. Using this observation, it is not difficult to adapt the proof of Theorem 2.8 to this case.*

The sharpness of the constants can be shown by considering examples in $C_0([0, 1])$.

One particular case of the above is used in the estimates considered by Bendaoud, Esterle and Mokhtari [2, 10].

Corollary 2.10 *Let $\gamma > 0$ and $0 < \alpha < \pi/2$. Let $(T(t))_{t \in S_\alpha}$ be an analytic non-quasinilpotent semigroup in a Banach algebra and let \mathcal{A} be the closed subalgebra generated by $(T(t))_{t \in S_\alpha}$. If there exists $t_0 > 0$ such that*

$$\sup_{t \in S_\alpha, |t| \leq t_0} \rho(T(t) - T(t(\gamma+1))) < k(S_\alpha),$$

with $k(S_\alpha) = \sup_{t \in S_\alpha} |\exp(-t) - \exp(-(\gamma+1)t)|$, then $\mathcal{A}/\operatorname{Rad} \mathcal{A}$ is unital and the generator of $\pi(T(t))_{t \in S_\alpha}$ is bounded, where $\pi : \mathcal{A} \rightarrow \mathcal{A}/\operatorname{Rad}(\mathcal{A})$ denotes the canonical surjection.

Now set $f_n(z) = z^n e^{-z}$, and set $k_n(\alpha) = \max_{z \in S_\alpha} |f_n(z)|$. A straightforward computation shows that $k_n(\alpha) = \left(\frac{n}{e \cos(\alpha)} \right)^n$.

If A is the generator of an analytic semigroup $(T(t))_{t \in S_\alpha}$, then we have $f_n(-tA) = (-1)^n t^n T^{(n)}(t)$. So the following result, which may be deduced from Hille's work, described in Theorem 1.1, means that if

$$\sup_{t \in S_\alpha, 0 < |t| < \delta} \|f_n(-tA)\| < k_n(\alpha)$$

for some $\delta > 0$, then the generator of the semigroup is bounded.

Theorem 2.11 ([1]) *Let $n \geq 1$ be an integer, let $\alpha \in (0, \pi/2)$ and let $(T(t))_{t \in S_\alpha}$ be an analytic semigroup. If*

$$\sup_{t \in S_\alpha, 0 < |t| < \delta} \|t^n T^{(n)}(t)\| < \left(\frac{n}{e \cos(\alpha)} \right)^n$$

for some $\delta > 0$, then the closed algebra generated by the semigroup is unital, and the generator of the semigroup is bounded.

The remainder of this section is devoted to quasinilpotent semigroups. We let $D(0, r)$ denote $\{z \in \mathbb{C} : |z| < r\}$.

Remark 2.12 *An analytic semigroup $(T(t))_{t \in S_\alpha}$ acting on a Banach space \mathcal{X} and bounded near the origin can be extended to the closed sector $\overline{S_\alpha}$. Indeed, assume that there exists $r > 0$ such that*

$$\sup_{t \in D(0, r) \cap S_\alpha} \|T(t)\| < +\infty.$$

Then $\lim_{\substack{t \rightarrow w \\ t \in S_\alpha}} T(t)x$ exists for every $x \in \mathcal{X}$ and every $w \in \partial S_\alpha$. Moreover if we set

$$T(w)x = \lim_{\substack{t \rightarrow w \\ t \in S_\alpha}} T(t)x,$$

then $(T(t))_{t \in \overline{S_\alpha}}$ is a semigroup of bounded operators which is continuous with respect to the strong operator topology. For we have $\lim_{\substack{t \rightarrow w \\ t \in S_\alpha}} T(t)T(t_0)x = T(t_0)x$ for every $t_0 > 0$ and every $x \in \mathcal{X}$. Now the result follows immediately from the fact that $\bigcup_{t > 0} T(t)\mathcal{X}$ is dense in \mathcal{X} , given that

$$\sup_{z \in D(0, r) \cap S_\alpha} \|T(z)\| < +\infty.$$

The next lemma demonstrates that nontrivial quasinilpotent analytic semigroups cannot be bounded on the right half-plane \mathbb{C}_+ . In fact, more is true.

Lemma 2.13 ([5]) *Let $(T(t))_{t \in \mathbb{C}_+}$ be a quasinilpotent analytic semigroup of bounded operators on a Banach space \mathcal{X} . Suppose that there exists $r > 0$ such that*

$$\sup_{t \in D(0,r) \cap \mathbb{C}_+} \|T(t)\| < +\infty,$$

and define $T(iy)$ for $y \in \mathbb{R}$ using Remark 2.12. If

$$\int_{-\infty}^{\infty} \frac{\log^+ \|T(iy)\|}{1 + y^2} dy < +\infty,$$

then $T(t) = 0$ for $t \in \mathbb{C}_+$.

In the case when the semigroup is bounded near the origin, we may give appropriate estimates on the imaginary axis.

Theorem 2.14 ([5]) *Let $(T(t))_{t \in \mathbb{C}_+}$ be a nontrivial quasinilpotent analytic semigroup satisfying the conditions of Remark 2.12, and let $s > 0$. Then*

$$\max(\rho(T(iy) - T(iy + is)), \rho(T(-iy) - T(-iy - is))) \geq 2,$$

for every $y > 0$.

From this we may obtain estimates for semigroups satisfying a growth condition near the imaginary axis.

Corollary 2.15 ([5]) *Let $(T(t))_{t \in \mathbb{C}_+}$ be a quasinilpotent analytic semigroup such that*

$$\sup_{y \in \mathbb{R}} e^{-\mu|y|} \|T(\delta + iy)\| < +\infty$$

for some $\delta > 0$ and some $\mu > 0$, and let $\gamma > 0$. Then

$$\sup_{t \in D(0,r) \cap \mathbb{C}_+} \|T(t) - T((1 + \gamma)t)\| \geq 2,$$

for every $r > 0$.

3 Lower estimates for functional calculus

In this section we summarise some very recent results from [6] and [7], which provide far-reaching generalizations of earlier work.

3.1 Semigroups on \mathbb{R}_+

3.1.1 The quasinilpotent case

Recall that if $(T(t))_{t>0}$ is a uniformly bounded strongly continuous semigroup with generator A , then

$$(A + \lambda I)^{-1} = - \int_0^\infty e^{\lambda t} T(t) dt, \quad (3)$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < 0$. Here the integral is taken in the sense of Bochner with respect to the strong operator topology. If, in addition, $(T(t))_{t>0}$ is quasinilpotent, then we have (3) for all $\lambda \in \mathbb{C}$.

Similarly, if $\mu \in M_c(0, \infty)$ (the space of complex finite Borel measures on $(0, \infty)$) with Laplace transform

$$F(s) := \mathcal{L}\mu(s) = \int_0^\infty e^{-s\xi} d\mu(\xi) \quad (s \in \mathbb{C}_+), \quad (4)$$

and $(T(t))_{t>0}$ is a strongly continuous semigroup of bounded operators on a Banach space \mathcal{X} , then we have a functional calculus for its generator A , defined by

$$F(-A) = \int_0^\infty T(\xi) d\mu(\xi),$$

in the sense of the strong operator topology; i.e.,

$$F(-A)x = \int_0^\infty T(\xi)x d\mu(\xi) \quad (x \in \mathcal{X}),$$

which exists as a Bochner integral.

The following theorem applies to several examples studied in [1, 9, 10, 14]; these include $\mu = \delta_1 - \delta_2$, the difference of two Dirac measures, where $F(s) :=$

$\mathcal{L}\mu(s) = e^{-s} - e^{-2s}$ and $F(-sA) = T(t) - T(2t)$. More importantly, the theorem applies to many other examples, such as $d\mu(t) = (\chi_{[1,2]} - \chi_{[2,3]})(t)dt$ and $\mu = \delta_1 - 3\delta_2 + \delta_3 + \delta_4$, which are not accessible with the methods of [1, 9, 10, 14].

Theorem 3.1 *Let $\mu \in M_c(0, \infty)$ be a real measure such that $\int_0^\infty d\mu(t) = 0$, and let $(T(t))_{t>0}$ be a strongly continuous quasinilpotent semigroup of bounded operators on a Banach space \mathcal{X} . Set $F = \mathcal{L}\mu$. Then there exists $\eta > 0$ such that*

$$\|F(-sA)\| > \max_{x \geq 0} |F(x)| \quad \text{for } 0 < s \leq \eta.$$

If $\mu \in M_c(0, \infty)$ is a complex measure, then we write $\tilde{F} = \mathcal{L}\bar{\mu}$, so that $\tilde{F}(z) = \overline{F(\bar{z})}$. By considering the real measure $\nu := \mu * \bar{\mu}$, we obtain the following result.

Corollary 3.2 *Let $\mu \in M_c(0, \infty)$ be a complex measure such that $\int_0^\infty d\mu(t) = 0$, and let $(T(t))_{t>0}$ be a strongly continuous quasinilpotent semigroup of bounded operators on a Banach space \mathcal{X} . Set $F = \mathcal{L}\mu$. Then there exists $\eta > 0$ such that*

$$\|F(-sA)\tilde{F}(-sA)\| > \max_{x \geq 0} |F(x)|^2 \quad \text{for } 0 < s \leq \eta.$$

3.1.2 The non-quasinilpotent case

Recall that a sequence $(P_n)_{n \geq 1}$ of idempotents in a Banach algebra \mathcal{A} is said to be *exhaustive* if $P_n^2 = P_n P_{n+1} = P_n$ for all n and if for every $\chi \in \widehat{\mathcal{A}}$ there is a p such that $\chi(P_n) = 1$ for all $n \geq p$. Such sequences may often be found in non-unital algebras: for example, $P_n = e_1 + \dots + e_n$ ($n = 1, 2, \dots$) in the Banach algebra c_0 .

Theorem 3.3 *Let $(T(t))_{t>0}$ be a strongly continuous and eventually norm-continuous non-quasinilpotent semigroup on a Banach space \mathcal{X} , with generator A . Let $F = \mathcal{L}\mu$, where $\mu \in M_c(0, \infty)$ is a real measure such that $\int_0^\infty d\mu = 0$. If there exists $(u_k)_k \subset (0, \infty)$ with $u_k \rightarrow 0$ such that*

$$\rho(F(-u_k A)) < \sup_{x>0} |F(x)|,$$

then the algebra \mathcal{A} generated by $(T(t))_{t>0}$ possesses an exhaustive sequence of idempotents $(P_n)_{n \geq 1}$ such that each semigroup $(P_n T(t))_{t>0}$ has a bounded generator.

If, further, $\|F(-u_k A)\| < \sup_{x>0} |F(x)|$, then $\bigcup_{n \geq 1} P_n \mathcal{A}$ is dense in \mathcal{A} .

3.2 Analytic semigroups

For $0 < \alpha < \pi/2$, let $H(S_\alpha)$ denote the Fréchet space of holomorphic functions on S_α , endowed with the topology of local uniform convergence; thus, if $(K_n)_{n \geq 1}$ is an increasing sequence of compact subsets of S_α with $\bigcup_{n \geq 1} K_n = S_\alpha$, we may specify the topology by the seminorms

$$\|f\|_n := \sup\{|f(z)| : z \in K_n\}.$$

We write $H(S_\alpha)'$ for its dual space; that is, the space of continuous linear functionals $\varphi : H(S_\alpha) \rightarrow \mathbb{C}$. This means that there is an index n and a constant $M > 0$ such that $|\langle f, \varphi \rangle| \leq M \|f\|_n$ for all $f \in H(S_\alpha)$.

We define the *Fourier–Borel transform* of φ by

$$\mathcal{FB}(\varphi)(z) = \langle e_{-z}, \varphi \rangle,$$

for $z \in \mathbb{C}$, where $e_{-z}(\xi) = e^{-z\xi}$ for $\xi \in S_\alpha$.

If $\varphi \in H(S_\alpha)'$, as above, then by the Hahn–Banach theorem, it can be extended to a functional on $C(K_n)$, which we still write as φ , and is thus induced by a Borel measure μ supported on K_n .

That is, we have

$$\langle f, \varphi \rangle = \int_{S_\alpha} f(\xi) d\mu(\xi),$$

where μ (which is not unique) is a compactly supported measure. For example, if $\langle f, \varphi \rangle = f'(1)$, then

$$\langle f, \varphi \rangle = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-1)^2},$$

where C is any sufficiently small circle surrounding the point 1. Note that

$$\mathcal{FB}(\varphi)(z) = \int_{S_\alpha} e^{-z\xi} d\mu(\xi).$$

Now let $T := (T(t))_{t \in S_\alpha}$ be an analytic semigroup on a Banach space \mathcal{X} , with generator A . Let $\varphi \in H(S_\alpha)'$ and let $F = \mathcal{FB}(\varphi)$.

We may thus define, formally to start with,

$$F(-A) = \langle T, \varphi \rangle = \int_{S_\alpha} T(\xi) d\mu(\xi),$$

which is well-defined as a Bochner integral in \mathcal{A} . It is easy to verify that the definition is independent of the choice of μ representing φ .

Indeed, if $u \in S_{\alpha-\beta}$, where $\text{supp } \mu \subset S_\beta$ and $0 < \beta < \alpha$, then we may also define

$$F(-uA) = \int_{S_\beta} T(u\xi) d\mu(\xi),$$

since $u\xi$ lies in S_α .

The following theorem extends [1, Thm. 3.6]. In the following, a symmetric measure is a measure such that $\mu(\overline{S}) = \overline{\mu(S)}$ for $S \subset S_\alpha$. A symmetric measure will have a Fourier–Borel transform f satisfying $f(z) = \tilde{f}(z) := \overline{f(\overline{z})}$ for all $z \in \mathbb{C}$.

Theorem 3.4 *Let $0 < \alpha < \pi/2$. Let $\varphi \in H(S_\alpha)'$, induced by a symmetric measure $\mu \in M_c(S_\alpha)$ such that $\int_{S_\alpha} d\mu(z) = 0$, and let $f = \mathcal{FB}(\varphi)$. Let $(T(t))_{t \in S_\alpha} = (\exp(tA))_{t \in S_\alpha}$ be an analytic non-quasinilpotent semigroup in a Banach algebra and let \mathcal{A} be the subalgebra generated by $(T(t))_{t \in S_\alpha}$. If there exists $t_0 > 0$ such that*

$$\sup_{t \in S_\alpha, |t| \leq t_0} \rho(f(-tA)) < \sup_{z \in S_\alpha} |f(z)|,$$

then $\mathcal{A}/\text{Rad } \mathcal{A}$ is unital and the generator of $\pi(T(t))_{t \in S_\alpha}$ is bounded, where $\pi : \mathcal{A} \rightarrow \mathcal{A}/\text{Rad}(\mathcal{A})$ denotes the canonical surjection.

By considering the convolution of a functional $\varphi \in H(S_\alpha)'$, with Fourier–Borel transform f , and the functional $\tilde{\varphi}$ with Fourier–Borel transform \tilde{f} , we obtain the following result.

Corollary 3.5 *Let $0 < \alpha < \pi/2$. Let $\varphi \in H(S_\alpha)'$, induced by a measure $\mu \in M_c(S_\alpha)$ such that $\int_{S_\alpha} d\mu(z) = 0$, and let $f = \mathcal{FB}(\varphi)$. Let $(T(t))_{t \in S_\alpha} = (\exp(tA))_{t \in S_\alpha}$ be an analytic non-quasinilpotent semigroup in a Banach algebra and let \mathcal{A} be the subalgebra generated by $(T(t))_{t \in S_\alpha}$. If there exists $t_0 > 0$ such that*

$$\sup_{t \in S_\alpha, |t| \leq t_0} \rho(f(-tA)\tilde{f}(-tA)) < \sup_{z \in S_\alpha} |f(z)||\tilde{f}(z)|,$$

then $\mathcal{A}/\text{Rad } \mathcal{A}$ is unital and the generator of $\pi(T(t))_{t \in S_\alpha}$ is bounded, where $\pi : \mathcal{A} \rightarrow \mathcal{A}/\text{Rad}(\mathcal{A})$ denotes the canonical surjection.

It is possible to obtain a similar conclusion, based only on estimates on the positive real line.

Theorem 3.6 *Let $0 < \alpha < \pi/2$. Let $\varphi \in H(S_\alpha)'$, induced by a symmetric measure $\mu \in M_c(S_\alpha)$ such that $\int_{S_\alpha} d\mu(z) = 0$, and let $f = \mathcal{FB}(\varphi)$. Suppose that $f(\mathbb{R}_+) \subset \mathbb{R}$. Let $(T(t))_{t \in S_\alpha} = (\exp(tA))_{t \in S_\alpha}$ be an analytic non-quasinilpotent semigroup in a Banach algebra and let \mathcal{A} be the subalgebra generated by $(T(t))_{t \in S_\alpha}$. If there exists $t_0 > 0$ such that*

$$\rho(f(-tA)) < \sup_{x > 0} |f(x)|,$$

for all $0 < t \leq t_0$, then $\mathcal{A}/\text{Rad } \mathcal{A}$ is unital and the generator of $\pi(T(t))_{t \in S_\alpha}$ is bounded, where $\pi : \mathcal{A} \rightarrow \mathcal{A}/\text{Rad}(\mathcal{A})$ denotes the canonical surjection.

The following example shows that the hypotheses of Theorem 3.6 are sharp.

Example 3.7 In the Banach algebra $\mathcal{A} = C_0([0, 1])$ consider the semigroup $T(t) : x \mapsto x^t$. Clearly, $(T(t))$ is not norm-continuous at 0.

For $x \in (0, 1]$ (which can be identified with the Gelfand space of \mathcal{A}) let $f = \mathcal{FB}(\mu)$ and

$$f(-tA)(x) = \int_{S_\alpha} x^{-t\xi} d\mu(\xi) = \int_{S_\alpha} e^{-t\xi \log x} d\mu(\xi),$$

where $\mu \in M_c(S_\alpha)$, supposing that $\int_{S_\alpha} d\mu(z) = 0$ and that $f(\mathbb{R}_+) \subset \mathbb{R}$.

Thus $f(-tA)(x) = f(-t \log x)$ and

$$\rho(f(-tA)) = \|f(-tA)\| = \sup_{x>0} |f(-t \log x)| = \sup_{r>0} |f(tr)|.$$

Clearly,

$$\sup_{t \in S_\alpha, |t| \leq t_0} \rho(f(-tA)) = \sup_{t \in S_\alpha} |f(z)|$$

for all $t_0 > 0$.

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